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JEL codes: N13, O3, O11

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Abstract

Allen (2009) has argued that the divergence in factor prices determined the direction of technical change that altered the course of economic growth in Britain. Using historical data for the 1700 – 1914 period, this paper derives and analyses the nature and direction of technical change. The results show that technical change was biased during the Industrial Revolution and that the bias stemmed from the divergence in the cost of labour and energy. In particular, labour saving responded strongly to the acceleration in wage growth in the 1850-1914 period. Overall, technical change was labour-saving, energy-using and hence capital-deepening. Moreover, the evidence shows that the expansion of effective energy supply allowed British economy to sustain output growth in the First Industrial Revolution era. Labour-saving innovations were particularly crucial in the Second Industrial Revolution.

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1. Introduction

Was technical change biased during the British Industrial Revolution? Did technical change respond to factor price changes? Allen (2009; 2011) argues that the technical change was biased towards saving labour, and that the bias stemmed from the divergence in the cost of energy and labour. This transformation affected the demand for technology by giving British businesses exceptional incentives to develop technologies that substituted capital and coal for expensive labour (Allen 2009, p. 137). The innovations were labour-saving, energy-using and hence capital deepening. Businesses were interested in reducing overall production costs[1], and the new technologies were just what they needed to adopt for that purpose.

Allen (2009) further states that the divergence in coal price and wages was a precursor for the industrialisation of Britain. Allen's (2009) explanation of the Industrial Revolution – linking the industrialisation of Britain to factor price developments – has been subject to criticism (Mokyr, 2009; Kelly *et al*., 2014). The critics state that the divergence in factor prices cannot explain the Industrial Revolution. Mokyr (2009) argues that Allen's (2009) model could be applied to only a few industries but not to the Industrial Revolution as a whole. He further claims that there was little evidence of labour-saving bias in technical change. Mokyr cites the evidence from patents compiled by Macleod (1988) that only 4.2% of all patents taken out in the 1660-1800 period had a labour-saving goal. Kelly *et al.*, (2014) suggest that the Industrial Revolution was a wave of technological advances that covered more than just textiles and iron. The new technologies were product-innovations as many of the inventions took the form of new and improved products. It is hard to classify all technological change to have factor-saving bias. However, Allen (2009) argues that price induced labour saving efforts were particularly intense in industries, which jointly accounted for the most of the productivity growth (i.e. textiles, mining, and iron).

The claim that the Industrial Revolution was a set of labour-saving, coal-using and capital deepening technological change at base could well be doubtful, but the existence of factor-saving bias in technical change during the industrial revolution period has previously garnered support from economic historians. von Tunzelmann (1994), for example, finds evidence in

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[1] High wage economy of Britain meant that the industrialists and entrepreneur innovators directed their efforts towards developing techniques that saved labour. Allen (2009) uses historical and cliometric accounts on cotton textile industry to demonstrate that the substitution of machines for labour reduced production costs significantly. For instance, Allen (2009, p. 208) argues that the aim of developing self-acting mules was to "*eliminate the jobs of the high wage spinners who had operated the mules, …*"

support of labour-saving bias in technical change after 1830s. Kander *et al.* (2013, p. 221-223) suggest that the take-off in labour saving took place between 1820 and 1830. Allen (2009), however, detects labour-saving bias as early as mid-eighteenth century. Somewhat in agreement with Allen (2009), Broadberry and Gupta (2009) document labour-saving innovations driven by high wages in cotton textiles to explain Britain's competitive advantage (in cotton textiles) over India in the eighteenth century.

So, did technical change respond to factor price changes? To find answer elements to this empirical question, this paper is set to assess the nature of the bias in technical change and its evolution, and to evaluate how labour saving efforts responded to movements in factor prices between 1700 and 1914. The paper derives a set of labour-augmenting and energy-augmenting technical change indices, and evaluates them with the aim of extending our understanding of how factor-saving innovations were integral to the industrialisation of Britain. Essentially, the objective is to corroborate Allen's (2009) conjecture, and to provide econometric evidence by extending and building on Allen's (2009) seminal work. A crucial factor in his analysis is the elasticity of substitution between the factors of production. Allen (2009) assumes there was a limited scope for factor substitution, however, he does not estimate the elasticity of factor substitution. This has serious implication for his empirical results, because the elasticity of substation is an important determinant of the direction of technical change (Acemoglu, 2002; Stern and Kander, 2012). Thus, this paper empirically estimates the elasticity of factor substitutions for the 1700-1914 period using maximum-likelihood procedure.

The elasticity of substitution is estimated assuming the economy's production technology is characterised by a general production function with a constant elasticity of substitution (CES) between capital/labour composite and energy. The CES function allows for the derivation of the time paths of two distinctive technology residuals, one augmenting labour and the other augmenting energy. Technical change is assumed to evolve according to an autoregressive process (AR). The AR process enables to account for the persistence of innovations and accumulation of technical knowledge. This assumption is supported by the tests of historical data on patent counts for unit roots. With the estimated elasticity of substitution at hand, the paper derives the implied indices of technical change i.e. factor-saving innovation indices. The derived (or implied) factor-saving innovation indices are interpretable, and the inspection of the technology indices jointly with factor prices and other aggregate data provides evidence about the nature of the technical change.

The empirical results reveal that technical change was biased toward saving labour and using energy between 1700 and 1914. More specifically, technical change responds strongly to changes in wages starting in the eighteenth century just as wages gradually rise. Efforts to save labour intensifies after the wage growth accelerates in around 1850. These observations suggest that British businesses appear to have directed their R&D efforts to save on labour as early as 1700. The paper interprets this finding as an evidence suggesting the presence of price induced technical change consistent with Allen's (2009) conjecture.

Energy-saving innovations closely track real coal price changes. As coal price drops over time, so does the index of energy-saving innovations until 1850. The index decelerates starting in 1850 in response to rising real coal price. This observation suggests that efficiency gains induced greater consumption of coal and hence energy-saving innovations appear to have been offset by the *rebound effects* – that is, energy efficiency improvements did not translate into equivalent reductions in energy consumption. Technical change was labour saving, but energy using and hence capital deepening. The evidence also indicates that labour-saving innovations were crucial in all periods but more so in the 1840-1914 period.

The next section reviews the historical background and provides a preliminary analysis of factor prices and other important variables. Section 3 briefly reviews the theory of induced technical change and outlines the aggregate production technology followed by a detailed discussion of the estimation method. The section ends with the derivation of the closed form solutions to the econometric problem. The subsequent section presents the results and the analysis of the derived technology residuals. Section 5 reconciles the econometric evidence with the historical accounts. The final section summarises the findings, and identifies topics for future research.

2. Induced technical change in the Industrial Revolution

This section outlines the historical accounts of some of the important factor-saving innovations, and provides an analysis of the price structure of the British economy[2]. The section reviews the trends in factor prices and in a number of factor-product and factor-factor ratios.

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[2] In what follows, terms innovation, technological progress and factor-augmenting technical change as well as factor-saving innovation are used interchangeably. These concepts generally refer to the efficiency term known as Solow-residual in neoclassical production technology.

2.1 Factor-saving innovations

Britain had a high wage economy long before the First Industrial Revolution owing to its commercial success in the world economy (Allen, 2009). Efforts to save labour were particularly prominent in the cotton textiles where Britain was to become the world leader in the nineteenth century. Workers were replaced by machinery, first in spinning, in printing, and later in weaving (Allen, 2009, 183-215). Two of the most important eighteenth-century technologies, Hargreaves's spinning jenny (1764) and Arkwright's water frame (1769), were important breakthroughs in cotton textiles. These two were subsequently combined into a new machine called the mule (1779) by Samuel Crompton. Adoption of these technologies, while reducing labour costs, did not have significant impact on average production costs in the early stages. Nevertheless, the new technology reduced demand for expensive labour: instead of one person working with one spindle, one person could supervise the operation of many spindles at once. By 1830, factory based manufacturing using these technologies reduced labour costs significantly. Labour costs declined from 17.19 d/lb of cotton in 1760 to 0.52 d/lb in 1836 (Allen, 2009, p. 185).

An important technology of the Industrial Revolution, steam engine, was invented several decades before it was first put to use successfully by Thomas Newcomen in the eighteenth century[3]. As an industrial technology, steam engine had not achieved economic significance until after it was used to drain mines in 1712. Newcomen engine was the first practical steam technology that generated economic value – miners were able to extract previously inaccessible coal reserves at reduced cost in large quantities[4]. This is an example of a factor-saving innovation.[5] Newcomen did not originally intend to economise on factor inputs. Instead, according to Kander *et al.*, (2013), his aim was to overcome the limits of power imposed by nature as animal power and manual labour had been used to drain coal mines. Animal fodder

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[3] See Dickenson (2012) for a summary of the development of steam technology

[4] Fouquet (2008, p. 78) appraises the contribution of steam technology to coal mining as follows "*Manual labour or animals could pump [water], but only if the quantities of water filling the mines were relatively modest. ... The introduction of the steam engine, burning cheap coal, managed to pump far greater quantities of water. Improvements in power technology had major impact on the production of coal*". A number of researchers dismiss the view the steam technology as a growth promoting factor in the early stages of industrialisation. Clark and Jacks (2007) provide a cliometric account of insignificance of steam technology to the expansion of coal supply. Crafts (2004) conducts growth accounting exercise to show that steam technology did not contribute to the growth of output prior to the second half of the nineteenth century.

[5] Allen (2009, p. 162) describes Newcomen engine as a biased technological improvement that shifted input demand away from an (expensive) animal feed and towards (cheaper) combustible fuel.

and labour were expensive in the eighteenth century, and coal was free at the pithead, therefore, the shift to coal-based steam technology reduced mining costs manifolds[6].

Another example of a factor-saving innovation is the case of the transition from expensive charcoal to cheap coal for iron smelting in the eighteenth century. Allen (2009) argues that the decline in the cost of iron smelting was the result of the relentless efforts to reduce the cost of iron smelting between 1709 and 1755. Charcoal was the main source of heat energy for iron smelting. Localised wood crisis raised the cost of wood in early eighteenth century (King, 2005). As a result, British ironworks lost competitiveness to Russian and Baltic suppliers. Cheap imported iron created a glut in the market leading to the closure of many ironworks between 1720 and 1740 (King, 2011). Prevailing high cost of charcoal eventually induced the shift to coke smelting. The shift not only reduced the energy costs, but also reduced the cost of labour. Between 1709 and 1850, the cost of labour per ton of pig iron declined from £1.50 to as little as £0.10 per ton (in 1755 prices) and the cost of energy declined from £17.50 to £3.56 per ton of pig iron (Allen 2009, p. 219). Thus, the shift to coke smelting reduced the cost of labour more than it reduced the cost of energy. Once again, British iron became competitive, and many new firms were established to take advantage of the new iron smelting method after 1760 (Allen, 2009, p. 218). The adoption of coke iron smelting was the result of an effective response to expensive charcoal smelting, and it is an example of a biased factor-saving innovation.

2.2 Factor prices, endowments and productivity

What was the nature of price structure in Britain? Allen (2009) provides a detailed account of the price structure in British economy for the 1600-1850 period. He puts emphasis on *unique* price structure of British economy characterised by *dear labour* and *cheap coal*. Allen (2009) traces the origins of high wages back to the fourteenth century when Black Death plague reduced the working age population significantly, and to Britain's success in international economy starting in the sixteenth century. By mid-eighteenth century, British nominal wages became relatively higher than those of other European workers (Broadberry and Gupta, 2006; Allen, 2001; Kander *et al.*, 2013). According to Broadberry and Gupta (2009), an unskilled labourer earned four to five times as much in Britain than in India, and the British unskilled

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[6] Natural outcome of the switch in the underlying source of energy enabled the miners to eliminate the expenditure on fodder. For the coal input at the pithead, more specifically, for coal residues, the miners did not have to pay no matter how energy intensive were the first Newcomen engines.

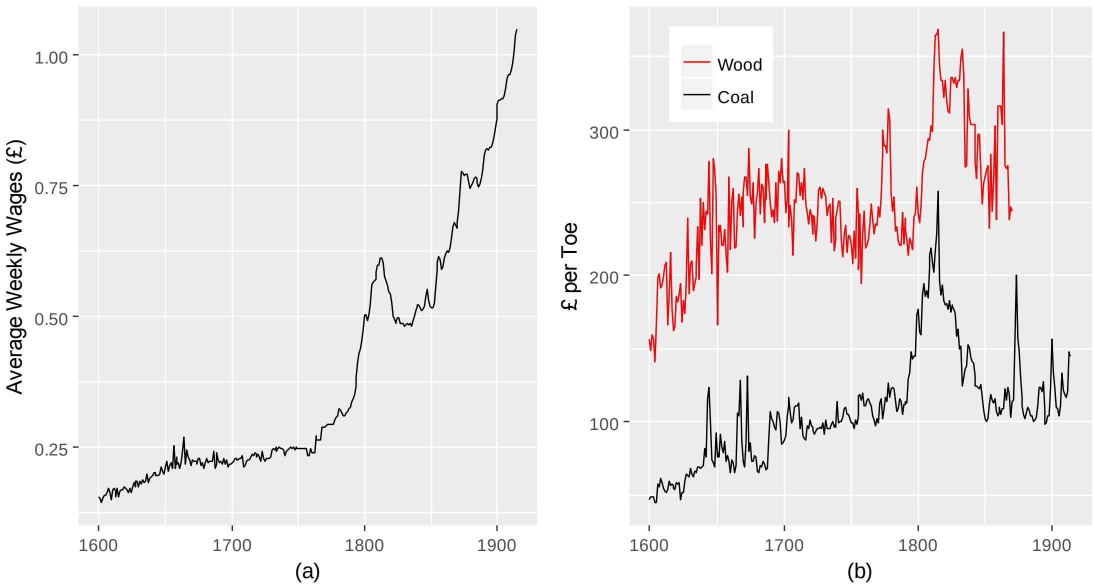

Figure 1 Average nominal wages and the cost of wood and coal, 1600-1914.

Panel (a): Average weekly wages. Panel (b): The price of wood (1600-1870) and coal in GBP per Toe.

silver wage during the second half of the eighteenth century was also less than twice as high as in Europe. British wages rose gradually for extended periods prior to the Industrial Revolution era. Wage growth increased drastically after 1760 as more and more labour shifted from agriculture to manufacturing and services, and to the growing urban centres (figure 1a).

Energy was cheap, but prices varied across Britain depending on the source and distance to woodlands or coal mining districts (Fouquet, 2008). For industrial purposes, woodfuels were primary choice, especially for fuel-intensive iron industry it was the only source of heat energy until the mid-eighteenth century. There was localised fuel scarcity in some regions as local woodlands had been depleted due to high demand from shipbuilders and rising demand for iron. Localised crisis meant wood had to be shipped from greater distances, and the cost of long distance transportation increased local woodfuel prices (Hammersely, 1973; Allen, 2003). Persistently high cost of woodfuels induced the shift to coke smelting in the eighteenth century. Yet, even after the shift, many ironworks continued using charcoal since charcoal iron smelting remained cheaper than coal smelting in some regions (Fouquet, 2008).

The steady growth of urban centres, especially of London, increased demand for fuel in the eighteenth century. High cost of transportation to towns and cities raised the cost of fuels to consumers over time. Nonetheless, coal was relatively cheap and, for some domestic needs, it was a substitute for woodfuels. Average price of woodfules was more than double the price of coal until early eighteenth century (Figure 1b). It was this gap between the prices lead to the

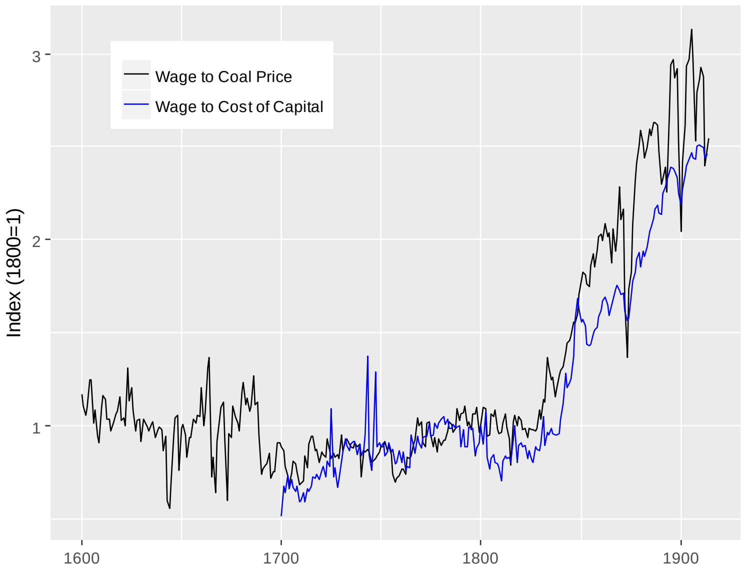

Figure 2 The ratios of average nominal wage to cost of capital and coal price, 1600-1914.

surge in coal demand, and hence to the all-out transition to coal in the eighteenth century (Allen, 2009). Cheap coal, however, was not an important source of industrial heat energy prior to 1750. As such, energy was not cheap as a whole, and efforts were primarily concentrated on saving energy (Fouquet, 2008). The gradual transition to coal from biomass, and the adoption of coal as the main source of heat energy played a significant role in bringing down energy costs. By the end of the eighteenth century, coal had become the dominant source of heat energy for domestic and industrial use.

Allen (2009) uses price ratios for several European cities to show that it made sense to invent and adopt labour-saving technologies only in Britain; the cost of labour (wages) relative to the cost of capital and energy was sufficiently high (to spur capital deepening), and this helps explain the technological innovations of the Industrial Revolution[7] (p. 138-144). Figure 2 plots the (nominal) cost of labour per unit of energy cost and per unit of rental cost of capital respectively[8]. There was no significant growth in nominal wages relative to other factor costs prior to 1830. The ratio of wages to coal price declines gradually until late seventeenth century before the trend reverses. From then onwards, the trend rises gradually until about 1830 and then accelerates drastically afterwards. The ratio of wages to the rental cost of capital has a

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[7] Allen's (2009) factor cost ratios are based on nominal price of coal in London and nominal wages of building labourers for the period between 1600 and 1850.

[8] Consistent with Allen's (2009) calculations, wages and prices are in money terms. Allen suggests that entrepreneurs and businesses considered actual (nominal) costs when adopting cost-saving technologies. Also, Allen (2009) uses the wages of building labourers only, here, the ratios is calculated using weekly average nominal wages.

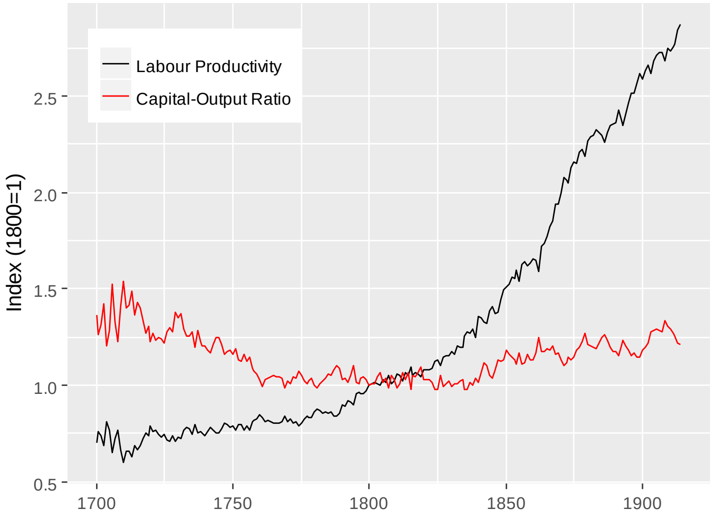

Figure 3 Labour productivity and capital-output ratio, 1700-1914.

broadly similar trend to that of the wage to coal price ratio. There was no drastic long-term divergence in the factor cost ratios prior to the nineteenth century. Nevertheless, this observation supports the narrative that energy and capital, as a bundle, gradually became cheaper relative to labour during the eighteenth century.

Did this transformation provide incentive for mechanisation? Allen (2009) argues that this was the precise reason why the Industrial Revolution was British. First, nominal wages were higher than both the cost of coal and capital. High wages were burden on businesses, and this lead to the development of labour-saving machines. Cheap coal coupled with low rental cost of capital offset the burden of high wages. This underpinned the gradual and persistent rise of wages as labour productivity increased in the eighteenth century. The persistent divergence in factor costs made mechanisation even more attractive and affected the employment of factor inputs in production through substitution of labour with energy and capital (Figure 2). This put the economy on a new, higher capital-labour ratio equilibrium where labour, endowed with more capital, increased productivity. High labour productivity meant persistent high wages, and high wages encouraged further labour saving. The productivity growth was the consequence of the race between technical progress and capital accumulation (Figure 3).

The data plotted in Figure 2 and 3 supports Allen's (2009) narrative. The growth of capital-labour ratio, a simple measure of mechanisation, remained low until the nineteenth century. The trend growth increases in the first half of the nineteenth century leading to a greater proportionate employment of energy and capital relative to labour. The bias was toward energy use and away from labour in a relative sense (Kander *et al.* 2013). British businesses began employing greater amounts of energy per unit of output and proportionately higher amounts of capital in production in the nineteenth century. Allen (2009) suggests these transformations were brought about by a handful of major innovations in textiles, iron and coal industries after the transition to coal. Kander *et al.* (2013) argue that the drop in the cost of cast iron was a significant turning point in the mechanisation of textiles. Although textiles utilised mostly water power prior to the 1830s, the metal required for making the machines became cheaper thanks to coke-iron smelting in the 1760s.

Figure 3 reveals that the capital formation per unit of output declined until 1760s, which indicates that economic activity was less capital intensive. After a period of stagnation between 1760s and 1830s, capital stock per unit of output began rising. Allen (2009) and Kander *et al.* (2013) point at the transition to cheap coal and the subsequent coal-based innovations for the reversal of the trend[9]. The shift to capital-intensive production reduced the reliance on expensive labour, and hence promoted labour productivity, especially so after the 1830s. The slow growth observed during the eighteenth century could be due to the gradual shift to steam power in only a small number of industries, most prominently in mining[10]. As the cost of steam power declined, the textile industry adopted the steam technology after 1830 boosting the productivity growth. This is consistent with von Tunzelmann's (1994, p. 289) conjecture on the existence of labour-saving technical change during the Industrial Revolution after 1830[11].

These historical observations clearly support the conjecture that technical change was biased towards saving expensive labour while expanding the use of the cheaper inputs, energy and capital during the Industrial Revolution. Moreover, Britain's industrialisation was not as fast in the eighteenth century as previously suggested[12]; and it was a gradual shift away from the traditional labour-intensive sectors to more energy intensive *modern* sectors during the eighteenth century. The shift was, clearly, dramatic in the nineteenth century.

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[9] See Fouquet (2008) for a useful systematic review of the energy technologies invented during the Industrial Revolution.

[10] Kanefsky and Robey (1980) estimate that, of the 2,191 steam engines built between 1700 and 1800, about 55% were in mining.

[11] However, von Tunzelmann (1994, p. 289) believes the marked rise in labour saving was eventually dampened by "*the continuing labour-surplus of males*".

[12] Dean and Cole (1962) provide the first estimates of output growth in Britain that paint a dramatic rise in productivity in the eighteenth century. Their estimates have received a wave of criticism by economic historians and new revised estimates by Harley (1982), Crafts (1985), Crafts and Harley (1992) and more recently Broadberry et al. (2011) show less than "dramatic" output growth in the eighteenth century. Griffin (2010) conducts a review of the scholarly work on the topic.

3. The model

The aim in this section is to describe the underlying theory for the model, present and discuss the production technology, and compute the two factor-augmenting residuals: labour-saving and energy-saving technical change indices[13].

3.1 Induced technical change theory

Allen (2009, p. 141) invokes Hicks' (1932) "*The Theory of Wages*" to reinforce his arguments. Hicks (1932) was the first to argue that a factor scarcity induced price changes determine the direction of technical change. He states, "*A change in the relative prices of the factors of production is itself a spur to invention, and to invention of a particular kind–directed to economizing the use of a factor which has become relatively expensive...*" (p. 124). In historical context, Rothbarth (1946) and then Habakkuk (1962) used Hicks' theory to analyse the direction of technical change in the US. Habakkuk (1962), in particular, argues that the abundance of land resources and the shortage of labour pushed up wages in the US in the nineteenth century. This in turn instigated the labour-saving innovations of American industrial revolution. Allen (2009, p. 15) sees eighteenth-century Britain as the prequel to nineteenth-century America. He argues that Britain's extensive coalfields played a similar role in the eighteenth century. Allen (2009), like Habakkuk (1962), limits his analysis to historical accounts and to descriptive analysis.

Related to Hicks' (1932) ideas, the 1960s saw the emergence of the induced technical change hypothesis. Fellner (1961) attempts to impose formal structure on the theory by defining two possible directions of technical change: labour-augmenting and capital-augmenting. Depending on the scarcity of factors, technical change could be directed towards one or the other. In the model, relative factor prices, $\frac{w}{r}$, determine the optimal relative factor ratio, $\frac{L}{K}$. If, for example, there is an excess demand for labour (labour scarcity case), the cost of labour (wage) increases forcing firms to invent labour-augmenting technologies. The reverse case drives the firms to increase the productivity of capital-augmenting technologies. Thus, the choice of innovation is dependent on relative factor prices.

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[13] The paper uses data on real GDP, energy prices, energy use, physical capital stock and cost shares of factors of production for the 1700-1914 period. Appendix A presents a detailed description of the data.

Fellner (1962) introduces two further concepts, quality and quantity of innovation. Quality of innovation characterises the direction of innovation whereas quantity of innovation refers to the extent of productivity growth. Because the innovations are allowed to increase average product of capital and average product of labour, the quality or the direction of innovation is then determined by the relative ratios of changes in average productivities of the input factors, $\frac{\Delta APL}{\Delta AKL}$. If the ratio is greater than unity, the innovation is labour-saving and vice versa. Fellner (1962) does not provide a fully developed model of induced technical change. However, it was the first attempt at developing a formal model of induced technical change. Fellner's (1962) idea has since been used to develop the theory of directed technical change by Acemoglu (1998; 2002) and others.

Later, Kennedy (1964) formalises the theory of induced technical change in a simple framework. The model economy has two inputs and the choice of innovation is not driven by relative factor prices, prices are assumed to be constant. The choice of innovation depends on the effectiveness of a certain innovation in reducing factor requirements for per unit of output. Therefore, the direction of technical change (or the choice of cost-reducing innovations) is determined by relative factor shares, and induced innovations pushes the economy to an equilibrium with a constant relative factor share (Acemoglu, 2002). The model is simple and introduces a novel idea of innovations possibilities frontier, and the model is not growth theoretic – there is no production function. The innovations possibilities frontier is independent of economic variables, that is, the frontier is exogenously given. Moreover, Kennedy's model does not spell out a microeconomic foundation of the behaviour of an innovating firm (Binswanger and Ruttan, 1978).

Later attempts at developing induced innovations models include Drandakis and Phelps (1966) and Samuelson (1965) who study the link between factor prices and technical change. However, the systematic study of the idea of Hicks (1932) that emphasised relative prices as the key determinants of the direction of technical change was not conducted until late 1990s and early 2000s[14]. Acemoglu (1998; 2000; 2002) reinvigorated the research on induced

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[14] There was little research on induced innovations hypothesis for almost 30 years. Eempirical studies on the topic for agricultural sector by Vernon Ruttan, Yujiro Hayami and Hans Binswanger are a notable exception. See Hayami and Ruttan (1970) and Binswanger and Ruttan (1978).

technical change theory, and introduced new features reformulating it as *directed technical change theory*.

Acemoglu (2002) formalises Habakkuk's (1962) arguments in a simple framework. Habakkuk (1962) argues that the intensity of efforts to save on labour was relatively higher in the US than in the UK in the nineteenth century. The adoption of labour-saving technologies by US businesses was driven by the fact that labour was relatively scarce in the US. The cost of labour was high but the country was endowed with land resources. The labour scarcity encouraged firms to develop and adopt labour-saving technologies and direct efforts towards energy using technologies through mechanisation. When Habakkuk's observations were brought into the modelling framework, Acemoglu (2002) finds that the elasticity of substitution between land and labour has to be less than one.

Recall that Allen (2009) likens Britain's path to industrialisation to that of the US, and assumes that the elasticity of substitution between labour and energy was probably less than unity. Allen (2009), however, does not provide an econometric estimate of the parameter. This leaves one to wonder whether Allen's historically elegant but empirically (and econometrically) untested conjecture holds when data is subjected to a formal econometric assessment. The next section introduces the econometric methodology used to estimate the factor elasticity parameter. From an empirical perspective, the econometric exercise conducted here is similar to that of Hassler *et al.* (2015) and to that of Hayami and Ruttan (1970) [15]. Hassler *et al.* (2015) investigate the nature of technical change induced by oil shocks of the 1970s in the US. Hayami and Ruttan's (1970) research probes the existence of price induced factor-saving technical change in the agricultural sector in the US and Japan.

3.2 Production Technology

The paper adopts the empirical methodology of Hassler *et al*. (2015). Following Hassler *et al*. (2015), the aggregate output is assumed to be produced using three factor inputs: labour,

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[15] In another similar study, Stern and Kander (2012) develop a one sector model of economic growth, and use the model to assess the role of energy in the industrialisation of Sweden. The present paper is similar to their study with respect to the production technology. They, too, use CES function with capital/labour composite and energy as input factors in empirical analysis. They, however, assess the role of energy in the transition to modern growth of Swedish economy. In contrast, the present paper studies the nature of technical change by deriving indices labour-saving and energy-saving technical change. The objective is to corroborate Allen's (2009) claim that the innovations of the Industrial Revolution were responses to high wages and cheap coal.

physical capital and energy. The production technology is assumed to be of a CES type as follows:

$$Y_t = \left[(1-\gamma)\left(\left(A_{l,t}L_t\right)^{\alpha}K_t^{1-\alpha}\right)^{\phi} + \gamma\left(A_{e,t}E_t\right)^{\phi}\right]^{\frac{1}{\phi}} \qquad 1$$

where L is labour, $A_{l,t}$ is the labour-augmenting technology, $A_{e,t}$ is the energy-augmenting technology. It is assumed that both the technology indices capture quality improvements in labour i.e. human capital and energy[16] over time. $\phi = \frac{\sigma-1}{\sigma}$, where, σ is the elasticity of substitution between capital/labour composite and energy services. γ measures the share of capital/labour composite and energy in production. The first argument in the production function is a Cobb-Douglas composite of capital and labour, and the second argument is energy services. Note that when the elasticity of substitution between factors of production is very large ($\sigma = \infty$), the Cobb-Douglas composite and energy are perfect substitutes, when $\sigma = 1$ the production function collapses to a Cobb Douglas function in all input arguments. When $\sigma = 0$ the Cobb-Douglas composite and energy are perfect complements, implying a Leontief function[17].

Is this specification empirically valid? Allen (2009) argues that mechanisation and the shift to factory based manufacturing during the Industrial Revolution meant that the scope for substitution between labour and capital broadened. For instance, power looms replaced hand looms in weaving, steam engines drained mines replacing manual work and mills were powered by steam engines. Therefore, the Cobb-Douglas specification for labour and capital appears to fit this narrative. CES function is widely used in studies using long time series data and in research on economic history (Stokey, 2001; Stern and Kander, 2012; Kander and Stern, 2014). Other forms of production technology have also been used in the analysis of technical change in economic history. Crafts (1985, 1994), Galor and Weil (2000), Lucas (2002), Hansen and Prescott (2002), Doepke (2004), Voigtlander and Voth (2006) used Cobb-Douglas specification. Such a specification is restrictive in assumption on elasticities of factor substitution. Cobb-Douglas technology requires the elasticity of factor substitution to be unity

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[16] Quality of energy reflects its productivity, flexibility, and other properties. Primary electricity is seen as the highest quality energy source and coal, wood, and other combustible biomass as the lowest, with oil, gas, and animal power at an intermediate level (Stern, 2010; Stern and Kander, 2012).

for all inputs. However, the scope for substitution between energy and other inputs is narrow. In such circumstances, the CES specification is particularly appropriate and useful in addressing the issue of biased technical change given that the function allows for the augmentation of multiple factors of production. Finally, this specification is statistically tractable and simpler to estimate empirically as fewer parameters have to be estimated (Stern and Kander, 2012).

As in Stern and Kander (2012), land is not a factor of production; instead, land is represented in the model as energy. In the pre-industrial economy, energy was mostly produced by the agricultural and forestry sectors that used land as an input. Recent evidence by Broadberry *et al.,* (2013) suggests that income share of agriculture in output shrank from 27% in 1700 to 19% in 1851. Fouquet (2008) shows that the income share of energy declined from 40% to 19% in the same period. Since the income shares of land and energy moved in the same direction, imposing a simplifying assumption on the model that land input is broadly represented by the energy input demonstrates the shrinking role of land and energy in production.

Here, the parameters of interest in this function are the two factor augmenting time varying technical change indices $A_{l,t}$ and $A_{e,t}$. In order to estimate the series, a Solow (1956) type exercise is performed below.

Partial elasticities of equation 1 with respect to labour and energy are

$$\frac{\partial Y_t}{\partial L_t}\frac{L_t}{Y_t} = \alpha(1-\gamma)\left[\frac{\left(A_{l,t}L_t\right)^{\alpha}K_t^{(1-\alpha)}}{Y_t}\right]^{\phi} \tag*{2}$$

and

$$\frac{\partial Y_t}{\partial E_t}\frac{E_t}{Y_t} = \gamma\left[\frac{A_{e,t}E_t}{Y_t}\right]^{\phi} \tag*{3}$$

Rearranging (2) and (3) and solving for the two technology trends gives

$$A_{l,t} = \left[\frac{Y_t}{K_t^{1-\alpha}L_t^{\alpha}}\left[\frac{\frac{\partial Y_t L_t}{\partial L_t Y_t}}{\alpha(1-\gamma)}\right]^{\frac{1}{\phi}}\right]^{\frac{1}{\alpha}} \tag*{4}$$

and

$$A_{e,t} = \frac{Y_t}{E_t}\left[\frac{\frac{\partial Y_t E_t}{\partial E_t Y_t}}{\gamma}\right]^{\frac{1}{\phi}} \tag{5}$$

Equations 4 and 5 indicate historical evolution of the two technical change series if values for σ, γ and α are known. α is the average value of estimated cost share of labour for the sample period, and is 0.6. γ determines the relative importance of factor inputs. In equations 4 and 5, γ is simply a scaling parameter, experimenting with different values shows that it plays the role of a level shifter only and the time path of innovations do not change. γ takes a value of 0.2 in the simulations. Finally, elasticity of substitution between capital-labour composite and energy, σ, is empirically estimated.

3.3 Elasticity of substitution

The substitution elasticity between the capital/labour bundle and energy determines the direction of technical change. It is empirically estimated by Maximum Likelihood Estimation (MLE) under the assumption that innovations evolve according to a random walk (with drift) process as follows:

$$a_t = \delta + a_{t-1} + \epsilon_t \tag{6}$$

here, $a_t = \log(A_t)$, δ is the drift parameter and $\epsilon_t \sim N\ (0, \Sigma)$. The assumption of random walk process for the arrival of innovations requires them to be correlated over time, and the drift term in equation 6 ensures that the process has a stochastic trend. These assumptions could be contested, but a strong support for this specification comes from historical data on granted patents. A unit root test of patents granted between 1700 and 1850 reveals that the series is a random walk with drift. Also, data on patent counts has been widely used in the literature as an indicator of innovative activity[18] (see Hall *et al.,* 2001; Madsen *et al,* 2010).

The use of patents in quantitative analysis of the Industrial Revolution has been criticised. Simple patent counts may not be a good measure innovative activity that made an economic contribution (Nuvolari, 2011). The contribution of some patented technologies may be insignificant while others such as Watt's steam technology created a turning point in the Industrial Revolution. Many improvements in production processes were not patented due to high the cost of patent protection (Moser, 2010). Allen (2009) suggests that many localised

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[18] Hall *et al.* (2001) developed a database of US patents, and the authors provide a detailed review of the advantages and disadvantages of using patent data as a measure of innovative activity in research in economics.

important small scale collective (i.e. micro) inventions in the mining and iron industries were not patented. In these industries, businesses shared knowledge of technical improvements without patenting them. Significant economic gains were realised using the unpatented inventions of many scientists during the era of the Industrial Revolution (Mokyr, 1993; 2009)[19]. Therefore, studies using patent data as a proxy for innovative activity may fail to fully account for the innovations outside the patent system. However, Moser (2010) suggests that high-quality innovations were more likely to be patented than average-quality innovations in the nineteenth century. Moreover, data on patents provide the best available information on innovative activities during the Industrial Revolution. Thus, conscious of the limitations, this data is used as an indicator of innovations[20].

Next, dividing equations 4 and 5 by their counterparts in period $t-1$ results in

$$\frac{A_{l,t}}{A_{l,t-1}} = \left[\frac{Y_t}{K_t^{1-\alpha} L_t^{\alpha}} \frac{K_{t-1}^{1-\alpha} L_{t-1}^{\alpha}}{Y_{t-1}} \left[\frac{\frac{\partial Y_t L_t}{\partial L_t Y_t}}{\frac{\partial Y_{t-1} L_{t-1}}{\partial L_{t-1} Y_{t-1}}} \right]^{\frac{1}{\phi}} \right]^{\frac{1}{\alpha}} \qquad 7$$

and

$$\frac{\mathrm{A_{e,t}}}{\mathrm{A_{e,t-1}}} = \frac{\mathrm{Y_t}}{\mathrm{E_t}} \frac{\mathrm{E_{t-1}}}{\mathrm{Y_{t-1}}} \left[\frac{\frac{\partial \mathrm{Y_t E_t}}{\partial \mathrm{L_t Y_t}}}{\frac{\partial \mathrm{Y_{t-1} E_{t-1}}}{\partial \mathrm{L_{t-1} Y_{t-1}}}} \right]^{\frac{1}{\phi}} \qquad 8$$

Taking logs of equations 7 and 8 and using them in equation 6 gives the following

$$\begin{bmatrix} \log \left(\frac{Y_t}{K_t^{1-\alpha} L_t^{\alpha}} \frac{K_{t-1}^{1-\alpha} L_{t-1}^{\alpha}}{Y_{t-1}} \right)^{\frac{1}{\alpha}} \\ \log \left(\frac{\mathrm{Y_t}}{\mathrm{E_t}} \frac{\mathrm{E_{t-1}}}{\mathrm{Y_{t-1}}} \right) \end{bmatrix} = \begin{bmatrix} \delta_{l,t} \\ \delta_{e,t} \end{bmatrix} - \frac{1}{\phi} \begin{bmatrix} \log \left(\frac{\frac{\partial Y_t L_t}{\partial L_t Y_t}}{\frac{\partial Y_{t-1} L_{t-1}}{\partial L_{t-1} Y_{t-1}}} \right)^{\frac{1}{\alpha}} \\ \log \left(\frac{\frac{\partial \mathrm{Y_t E_t}}{\partial \mathrm{L_t Y_t}}}{\frac{\partial \mathrm{Y_{t-1} E_{t-1}}}{\partial \mathrm{L_{t-1} Y_{t-1}}}} \right) \end{bmatrix} + \begin{bmatrix} \varepsilon_{l,t} \\ \varepsilon_{e,t} \end{bmatrix} \qquad 9$$

In vector notation, equation 9 can be expressed as

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[19] Mokyr (2009a, p. 353) argues that the patenting system had side effects on the level of inventive activity. Mokyr, referring to inventors, states that "… *the patent system, for the vast majority of them, offered a false hope, and the expected payoff of a patent was in all likelihood negative*".

[20] Alternatively, innovative activities can be measured using R&D expenditure. For the sample period 1700 - 1914, data on investment in R&D is not available. See Cameron (1996) for a survey of literature on various measures of innovation used in empirical and theoretical studies.

$$\boldsymbol{y_t} = \delta - \frac{1}{\phi}\boldsymbol{x_t} + \epsilon_t \qquad 10$$

Assuming $\epsilon \sim N\ (0, \Sigma)$, the log-likelihood function is then given by

$$l(\boldsymbol{y}|\delta, \phi, \Sigma) = -\frac{N}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=1}^{N}\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right)^T \Sigma^{-1}\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right) \qquad 11$$

The derivation of the closed form solutions to this problem is presented in Appendix B.1.

4. Results

$\hat{\boldsymbol{\sigma}}$	$\hat{\boldsymbol{\delta}}_l$	$\hat{\boldsymbol{\delta}}_e$
0.185 (0.016)	0.007 (0.03)	-0.007 (0.034)

Table 1 Parameter estimates.

Bootstrap standard errors are in parenthesis. The standard errors are baesd on 20,000 simulations.

The estimated elasticity of substitution is 0.185; it is close to zero and statistically significant. First, this implies that the labour/capital composite and energy are complements, thus there was a limited scope for factor substitutability, at least in the short run. Second, given that the elasticity of substitution is less than unity, there was some degree of bias in technical change towards using more abundant factor and saving the scarce input. In the present context, as shown in section 2, the technical change was possibly directed towards saving the expensive (scarce) input, labour, and towards increased use of more abundant factor, energy (coal). This econometric evidence, as will be shown in further analysis, supports Allen’s (2009) conjecture.

4.1 The elasticity of substitution

The low elasticity of substitution reflects an important feature of the British Industrial Revolution. Britain had an energy economy, the share of energy expenditure exceeded 20% in the First Industrial Revolution, the growing stock of capital required greater amounts of energy to be operational. Investment in plant, machinery and equipment increased by more than 10 times its original value between 1760 and 1850, and a large proportion of investment was in buildings and works (Feinstein and Pollard, 1988, p. 446). Under such circumstances there was little room for substitutability between energy and capital. From a theoretical perspective, the estimated low elasticity of substitution implies that both the capital/labour bundle and energy must increase in tandem for growth to occur[21]; There needs to be fixed proportions of factor inputs to maintain growth in output, a case of Leontief production technology. More capital/labour bundle with limited energy supply would have pushed growth back to levels seen during the pre-industrial periods (Wrigley, 2010). Likewise, abundant supply of energy without sustained capital accumulation would not have supported economic growth.

The technology drift coefficients $\boldsymbol{\hat{\delta}_l}$ and $\boldsymbol{\hat{\delta}_e}$ are 0.007 and -0.007 respectively. The estimates are similar in magnitude but they have different signs implying that the technology indices drift away from each other in time. Again, combined with low elasticity of substation, this finding supports Allen's (2009) conjecture that the technical change was biased during the Industrial Revolution. Using the parameter estimates (from table 1) in equations 3 and 4, the time paths of the two factor-saving innovations are computed and presented in figure 5. The series are normalised to one in 1800 for ease of trend analysis.

4.2 The technical change indices

In this section, the technology indices, A_l and A_e, are examined to ascertain the nature and behaviour of technical change. Allen (2009) argues that the crucial technological innovations of the Industrial Revolution were a response to high wages and cheap coal. Do the empirical results corroborate Allen's (2009) claim? Figure 2 shows the time paths of implied labour-saving and energy-saving technology indices alongside average real wages and coal price. Clearly, technology indices track factor prices closely and appear to respond to changes in factor prices. For instance, a break in the trend of the real wage index in 1750 is followed by a trend break in the labour-saving innovations index in around 1775. Then, significant trend breaks occur between 1850 and 1875, first in real wages and then in labour-saving innovations index. Between the two sets of breaks, there is a turning point in the trend of the labour-saving innovations in 1820s. Overall, there was a labour-saving bias well before the onset of the Industrial Revolution – labour-saving technology index grows gradually between 1700 and 1770 at 0.32% per annum. Labour saving intensifies marked after about 1850, most likely, in

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[21] Given that the elasticity of substitution is close to zero, the production function is approximately a Leontief function, and the capital/labour composite and energy are complements.

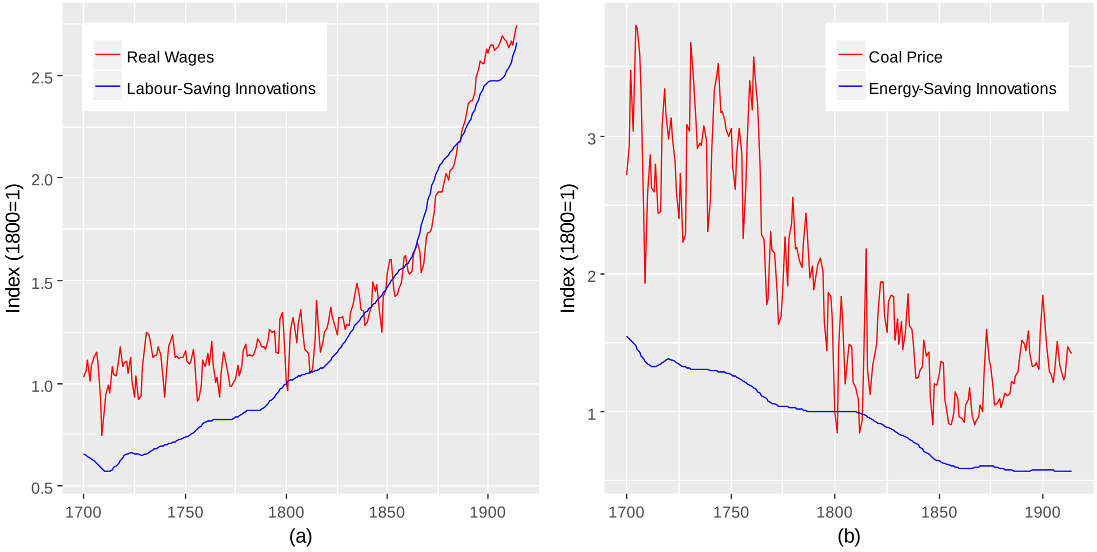

Figure 2 Time paths of factor-saving innovations and factor prices.

Panel (a): Real wages and labour-saving technical change index. Panel (b): Real coal price and energy-saving technical change index. Factor-saving technical change indices are smoothed using Hodrick-Prescot filter to smooth out cyclical fluctuations.

response to wage growth. This evidence goes against von Tunzelmann's (1994) narrative that British economy did not experience labour-saving technical change until 1830. Labour saving was present as early as 1700s, and, in agreement with Allen's (2009) narrative, labour-saving innovations may have been responses to growing wages.

Energy-saving innovations index declines continuously tracking the changes in coal price but with a time lag. Each downward movement in coal price is either followed or preceded by a gradual decline in the technology index. After persistently dropping, real coal price stops declining in 1850s, and begins rising in around 1875. The energy-saving innovations appear to respond strongly to this by stagnating. Does the continuous decline in the energy-saving innovations index suggest there was no energy saving technological change during the Industrial Revolution? No, the continuous downward movement in A_e simply reflects the diminishing efforts to save energy. To put it differently, the index reveals growing efforts to use more energy over time as coal price declines. This will be examined in more details in the next section. Similarly, continuous rise in A_l, the labour-saving innovations index, indicates the growing efforts to save labour as average wages rise.

How strongly did technical change respond to prices? This requires a closer inspection, therefore, average annual growth rates of technology indices and factor prices are computed for the pre-industrialisation (1700-1770), the First Industrial Revolution (1770-1840) and the Second Industrial Revolution (1840-1914) periods as well as for the full sample (1700-1914).

	1700 – 1770	**1770 – 1840**	**1840 – 1914**	**1700 – 1914**
	Technology Indices			
Energy-Saving Innovations (A_e)	-0.53%	-0.48%	-0.39%	-0.47%
Labour-Saving Innovations (A_l)	0.32%	0.71%	0.92%	0.66%
Ratio $\frac{A_l}{A_e}$	0.94%	1.22%	1.26%	1.14%
	Factor Prices			
Wage (P_l)	0.13%	0.20%	1.02%	0.46%
Energy Price (P_e)	-0.32%	-0.78%	0.19%	-0.30%
	Correlation			
Correlation (A_e, A_l)	-0.82	-0.95	-0.76	-0.90
Correlation (A_e, P_e)	0.36	0.30	-0.06	0.86
Correlation (A_l, P_l)	0.20	0.70	0.98	0.96

Table 2 Average growth rates of key variables and correlation analysis

Table 2 presents the growth rates and correlation coefficients for the variables of interest. There is a gradual built up in the efforts to save labour in the 1700-1770 period while average real wage, P_l, remains high but stagnant. As average wage grows gradually, the growth rate of A_l increases from 0.31% in the pre-industrial period to 0.72% in the First Industrial Revolution period. This is a strong response in labour-saving efforts to wage changes; the strongest response of A_l is registered in the Second Industrial Revolution when the growth of wages, P_l, and A_l accelerates. The correlation between them is strong giving a coefficient of 0.98. Efforts to save labour intensifies during the apex of the industrialisation. This coincides with the revisionist historians' much-debated productivity growth acceleration of the nineteenth century.

The response of energy-saving technical change to coal price changes appears to be time lagged. Therefore, the analysis of the growth rates may suggest a weaker response. However, the response was stronger than is revealed in the data because the persistent drop in the real price of coal began in early 1600, and hence precedes the response of energy-saving innovations of 1700 by at least 100 years[22]. The correlation analysis shows that A_e responds relatively strongly to changes in real coal price in the 1700-1840 period, when the price of coal, P_e, declines by 54%. The reversal of the trend of coal price in around 1875 appears to put an

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[22] Real price of coal for earlier periods (1600-1699) is not presented here to maintain consistency.

end to the long downward movement of A_e. Equally striking is the growth rate of relative innovations (the growth rate of ratio $\frac{A_l}{A_e}$) and the correlation between the indices. The growth is positive in all sub-periods and it increases in magnitude over time. Labour-saving efforts dominate in all periods, especially so in the 1840-1914 period. Moreover, the correlation between the factor-saving innovations is negative and strong in all periods, which again, may reflect the strong bias in technical change.

Overall, the findings show that there was, initially, a gradual movement towards the development of labour-saving, capital-intensive and resource-using production technology. The efforts then appear to have intensified in the labour-saving direction in the Second Industrial Revolution, which brought about the acceleration in productivity growth as revisionist historians have pointed out (Crafts and Harley, 1992)[23]. However, the evidence may appear to contradict the conventional wisdom that energy saving-innovations increased during the Industrial Revolution. As will be seen in the next section, energy-saving efforts did not slow down, instead, innovations in energy technologies induced greater consumption of energy. Thus, the decline in the energy-saving innovations index reflects the *rebound effects* arising from energy-saving innovations.

To put this in the context of the Industrial Revolution, efforts to save labour meant greater efforts to use energy as labour was replaced by machines. Incremental energy saving innovations resulted in greater demand for cheaper energy and mechanisation became more attractive. The ultimate effect was the sustained increase in productivity growth, especially in the Second Industrial Revolution. Thus, labour saving was the indirect outcome of the energy-saving efforts, which enabled the expansion of energy service provision.

5. Discussion

The aim of this section is to discuss the evidence presented in the preceding section in the context of the economics of the Industrial Revolution drawing on the accounts and narratives of economic historians, and to reconcile the empirical results with the historical accounts. It is important to evaluate the quantitative evidence discussed in the preceding section in historical context. Given the historical nature of the data used in this study, it is difficult, for instance, to

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[23] See Griffin (2010) for a detailed discussion of historical revisions of output and TFP series.

explain the dynamics of factor-saving technologies without explaining the context within which they were observed.

An important finding of this study is that the technical change indices closely tracks the long run trends in real coal price and average real wage. Results presented in the preceding section show that energy-saving innovations declined during the Industrial Revolution era. Why did energy-saving innovations decline? Recent research documents that economic energy efficiency increased many folds in the 1700-1914 period[24]. Energy transitions from biomass to coal, to liquid fuels and then to electricity undoubtedly stimulated economic energy efficiency. Improvements in the thermal efficiency of machines led to further innovative activities. Fouquet (2008) discusses the various important energy-saving technologies, their practicality, social and economic significance. These include improvements in the utilisation of steam power in a range of industry sectors and increased efficiency of iron furnaces and forges, the use of high pressure steam engines in railroads, shipping and passenger transport services. In the following, attempts are made to reconcile the observed decline in energy saving with the historical accounts.

5.1 Energy-saving and rebound effect

Technical change could have a double effect on energy; it could be energy-saving and/or energy-expanding (Kander *et al.*, 2013). Thus, by considering the existence of the *rebound effect* over the sample period, it is possible to explain why energy-saving innovations index has a downward slope. The rebound effect arises from increased demand for energy, mainly, but not entirely, due to persistent declines in energy prices. Generally, a greater rise in energy intensity than the fall in energy prices would suggest a *rebound effect* (Fouquet, 2008, p. 277). Besides, energy-saving innovations bring about further effective reductions to the cost of energy services provision, and hence could boost the demand for energy.

As shown in Table 2 and 3, energy consumption growth increased over time whilst its price declined. The rate of growth of energy consumption is much higher than the rate of decline in average energy price, indicating the high elasticity of demand for energy. Moreover, the growth of energy intensity outpaced the rate of decline in energy price in the First and Second Industrial Revolution. It is possible that the increased energy consumption offset a greater fraction of

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[24] See Fouquet (2008) and Kander *et al.* (2013).

	1700 – 1770	**1770 – 1840**	**1840 – 1914**	**1700 – 1914**
Labour (L)	0.44%	0.89%	1.00%	0.79%
Effective Labour ($A_l * L$)	0.76%	1.60%	1.92%	1.45%
Energy (E)	1.09%	2.06%	2.34%	1.85%
Effective Energy ($A_e * E$)	0.55%	1.58%	1.94%	1.37%
Energy Service Price ($A_e * P_e$)	-0.34%	-0.60%	-0.65%	-1.02%
Energy Intensity	0.30%	0.53%	0.37%	0.37%
Capital-Labour Ratio ($\frac{K}{L}$)	-0.20%	0.71%	1.27%	0.60%
GDP per capita	0.29%	0.44%	1.03%	0.59%

Table 3 Average growth rates of factor inputs and output per capita

energy efficiency improvements during the Industrial Revolution[25]. The existence of the *rebound effect* stemming from energy savings during the Industrial Revolution was acknowledged by William Stanley Jevons as far back as 1865 (Madureira, 2012). In his book, The Coal Question, Jevons states that the economical use of fuel should be equivalent to the diminishing consumption of it, but he observes that the contrary was true. He points out that the energy efficiency gains are followed by many-folds increased consumption of coal in the iron industry in Scotland between 1830 and the 1860s (Jevons, 1865; chapter 7).

In fact, *rebound effect* from energy-savings was observed even before 1830 in iron industry. By the end of the seventeenth century, the cost of iron smelting increased, and imported iron was relatively cheap. In order to compete with foreign suppliers, British iron suppliers had to reduce the cost of iron. Iron making is an energy intensive process; thus, fuel costs were the major part of total costs (Hammersely, 1973; Fouquet, 2008). Sustained experiments to substitute charcoal with a cheaper alternative, coal, resulted in a practical energy-saving innovation. Transition to coke smelting reduced energy cost as coal was relatively cheaper than charcoal, but it did not lead to saving energy as was originally intended. Instead, its indirect effect was a rapid increase in coal consumption driven by high demand for bar iron. Cheap coal meant cheap metal, and cheap metal reduced the cost of capital equipment encouraging mechanisation of production. In consequence, demand for coal exploded in the Second Industrial Revolution.

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[25] Kander *et al.* (2013, p. 30-32) present a useful summary of the effects that technical change in energy technologies could bring about.

The *rebound effect* was not limited to the iron industry. Fouquet (2008 p.276-279) documents a range of historical accounts of the *rebound effect* in the provision of various energy services. These include transport services provision, lighting and heating. For instance, freight transport services became cheaper due to the application of steam technology in sea and rail transport service provision in the nineteenth century. The desire for timely and safe delivery of goods over long distances at low cost stimulated the demand for freight transport services. The demand for passenger transport services increased as a result of falling passenger transport service prices. Similarly, the advances in lighting technology led to the reduction in the cost of lighting during the Industrial Revolution era as documented in Fouquet and Pearson (2012). The net effect of these changes resulted in a spectacular rise in the demand for coal, gas and kerosene (for lighting).

5.2 Energy expansion, capital deepening and the cost of capital

Energy-saving innovations, although declined, appear to have had important effects on overall productivity growth. Implied correlation coefficient between the two factor-saving innovations is -0.90; it offers strong support for the existence of directed (or induced) technical change. As seen in Table 3, average growth rate of labour supply declined in the Second Industrial Revolution whilst demand for energy and capital increased. In particular, energy-saving innovations continuously declines at an average annual rate of 0.39% between 1840 and 1914. In the same period, the consumption of energy increases at the highest rate of 2.34% per year. Capital-Labour ratio grows at an average annual rate of 1.27% whilst labour supply grew at an annual rate of only 1.00%. These observations suggest that businesses directed their efforts towards saving labour and using greater amounts of energy and capital[26].

Incremental advances in energy saving (i.e. gradual perfection of steam engines), and the decline in the cost of energy reduces the cost of energy services (i.e. effective energy price in Table 3) and promotes the expansion of the production possibilities frontier. That is, cheap energy coupled with the energy-saving innovations enables greater accumulation of capital stock. In turn, the scale of production requires even more energy to sustain the productivity growth. The net result is the mechanisation of production reduces labour demand, and hence increases output per labour input. In line with productivity growth, wages grew faster and

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[26] Kander *et al.* (2013, p. 219-231) provide a brief summary of the nature of factor-saving innovations during the Industrial Revolution. They note that labour-saving innovations resulted in growing demand for energy driven by the expansion of capital intensive industries in the Industrial Revolution.

labour-saving efforts rapidly increased after the 1850s. Because of this self-perpetuating loop, British businesses did not run into diminishing marginal returns to inputs. Together with labour-saving innovations, energy-saving innovations ensured the sustained accumulation of capital[27].

Allen (2009) and Kander *et al.* (2013), studying trends in raw data, make a case for the possibility of bias in a capital deepening (accumulation) direction during the Industrial Revolution. More specifically, capital deepening does not occur in isolation; building capital equipment requires cheap raw materials i.e. bricks, metal, energy. The production of low cost raw materials (i.e. construction materials) requires cheap energy. Once built, capital stock requires even more energy input to have economic value. As such, capital deepening must be coupled with energy expansion at every stage of capital accumulation. The diffusion of steam technology during the Industrial Revolution could lend support to this conjecture. A steam engine, especially in its primitive form, was not made of only cast iron, but also of bricks, wood and iron among other materials. Metal parts were around 60% of the total cost, the rest of the materials making up the remaining cost[28]. As the transition to coal was well underway, all these raw materials became cheaper, especially iron. The reduction in the cost of acquiring physical capital, and further tinkering with the steam technology for efficiency gains reduced the effective cost of providing power. Fouquet (2008, p. 120) estimates that the running cost of generating one kWh of power was around 450 pence in 1760. The cost fell to 100 pence by 1800 and to 30 pence by 1870. This transformation was key to the widespread diffusion of steam technology and hence the greater accumulation of capital boosting demand for energy.

5.3 Mechanisation, energy and skilled labour, 1850-1914

The period between 1850 and 1914 requires careful analysis and discussion. There is notable change in the trend growth of both innovations. Average annual growth rate of labour-saving innovations is 0.92%, higher than observed in previous periods, and energy-saving innovations index declines by 0.17% per year, lower than observed in preceding periods. The paper has interpreted these behavioural shifts as strong responses to increases in factor prices. In what

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[28] Fouquet (2008, p. 119) documents changes in the cost of building and running a Newcomen engine.

follows, alternative historical accounts are discussed to shed further light on the behaviour of the energy-saving technical index.

Why did the energy-saving innovations stagnate after 1850? According to von Tunzelmann (1994, p. 287) once coal became a common fuel of choice for industrial purposes, businesses had to find ways to cut energy costs. If in the early stages of industrialisation innovations were coal using, the situation reversed in the latter stages, and greater efforts were directed towards developing coal-saving innovations. In particular, the switch to coal in the early stages of industrialisation removed the logistical and supply challenges associated with using expensive charcoal. A new source of energy, coal, was elastically supplied and was available in abundant quantities at relatively low cost. This in turn instigated efforts to develop coal-using technologies. These innovations enabled businesses to achieve scale economies, which boosted the demand for coal in subsequent decades. The thirst for coal eventually made it the basis of the economy’s fuel supply by the mid-nineteenth century. From then on, the greater innovative activities were directed at economising on coal use[29]. Being a common fuel for industrial use, coal made up a sizeable fraction of production costs; thus, efforts were directed at reducing energy costs eventually[30].

Some of the observed changes in factor-augmenting innovations in the 1850-1914 period could have been due to, and sustained by, the changes in the quality of energy. It is possible that the transition to liquid fuels and electricity decelerated the decline in energy-saving innovations, and encouraged further efforts to save labour save. For instance, commercial use of electric power allowed greater flexibility in the design of production processes. By the 1880s, electricity powered motors were attached to individual machines. Previously, the manufacturing process was designed around a single engine generating all the power transmitted to the machines via shafts (Devine, 1983). While electricity using power generation reduced the energy required to drive the machinery, its adoption involved major changes in factory design and machine organisation. These dramatic changes enabled new industries to obtain greater output per unit of capital and labour input.

In the second half of the nineteenth century, educational attainment of the population rapidly increased in Britain (de Pleijt, 2015). Britain’s high wage economy permitted to improve the

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[29] See von Tunzelmann (1994, p. 287) for a summary of the major nineteenth century fuel-saving innovations. For a more detailed discussion refer to Fouquet (2008).

[30] Fouquet (2008, p. 89) notes that in the iron industry, the desire for energy costs to enable competition with the foreign suppliers led to the invention of more efficient ways of producing iron and in particular steel in the 1850s.

quality of labour through greater educational attainment of workers (Allen, 2011). Technological progress accelerated after the 1830s, and it was mostly driven by scientific discoveries (Mokyr, 1990; 2009). The new innovations were possibly skill-biased as technologies became more advanced and complex towards the end of the nineteenth century. Advances in technologies increased the relative demand for skilled and educated workers, and educational advance increased their relative supply (Goldrin and Katz, 2008). Just as the energy quality is reflected in the energy-saving innovations index, estimated labour-saving innovations account for the rise in the quality of the labour force i.e. rise in skill-set and education of workers. Therefore, the observed sharp rise in labour-saving innovations, in the last quarter of the nineteenth century, could also be explained by the increased educational attainment of workers. Mokyr (1990; 1993) goes as far as claiming that the Second Industrial Revolution was sustained by the efforts of engineers, scientists and other people with formal training and education and that it was a scientific revolution.

6. Conclusion

This paper extends Allen’s (2009; 2011) cliometric analysis to provide econometric evidence on the nature and the importance of technical change during the Industrial Revolution era. The purpose of this study is to extend our understanding of how factor-saving innovations were integral to the industrialisation of Britain. Using historical data from 1700 to 1914, the paper derives energy-saving and labour-saving technical change indices. With the added assumption that the innovations follow a specific data generating process, the elasticity of substitution between labour/capital bundle and energy is estimated using maximum-likelihood procedure. With the estimated elasticity of factor substitution, the indexes of energy-saving and labour-saving innovations are derived. The analysis of the implied technical change series shows that the technical change responded to changes in wages and energy prices consistent with Allen’s (2009) seminal findings.

What was the nature of the bias? The bias was the result of technical change responding to divergence in factor prices. Real wages continuously increased in the 1700-1914 period, and real energy price persistently declined. British businesses responded to growing wages and declining coal prices by investing in labour-saving but energy-using technologies. The correlation between the factor-saving technical change and factor prices indicates that the strongest response to factor prices was in the Second Industrial Revolution.

How important was induced technical change to the industrialisation of Britain? It was important in many ways. First, energy-saving innovations increased the demand for energy thereby expanding the market for coal. Coal was abundant, and it was a cheap input into production, which meant that Britain did not face obstacles in its path of industrialisation as its continental counterparts did. Second, energy-saving innovations reduced the cost of capital – cheap coal implied cheap metal, and hence low cost machines. Greater investment in machinery coupled with the cheap energy shifted production from homes to factories resulting in labour savings. Labour saving was sustained only because it was possible to expand energy use. Eventually, ensuing productivity growth sustained higher wages, which improved the living standards of workers in the nineteenth century. It was then possible to improve the quality of British labour through education and training. A highly skilled workforce was pivotal to the productivity growth seen in the 1850-1914 period as suggested by Allen (2009).

The results of this paper reinforce our understanding of the Industrial Revolution as characterised by Allen (2009). In this respect, Britain's industrialisation was possibly a result of a response to pressures arising from market conditions whereby labour was expensive and energy became cheap. Above all, Britain's coal reserves played a major role in its escape from the low-growth economic system. The empirical results are, however, highly sensitive to data revisions and limitations. A major limitation of this paper stems from the speculative nature of capital stock data for the 1700-1760 period. Nevertheless, future revisions and assimilations of historical data may permit to extend the present study. One fruitful avenue for future research is to use sectorial data to assess how technical change responded to price movements in different sectors. As suggested by the critics of Allen's conjecture, labour saving was not widespread and possibly limited to a handful of industry sectors.

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Appendix A

This appendix provides a detailed description of the sources and the construction of the data used in the thesis. The data is collected from various independent sources and, most of the data is available for the whole sample period from 1700 to 1914. For the econometric analysis, some series are interpolated, spliced and/or extrapolated. All transformations are made under plausible assumptions by drawing on relevant literature.

A.1 Data sources and methods

GDP and GDP Deflator

Data on Nominal GDP and GDP deflator are available from Broadberry *et al.* (2015).

Labour

Labour input used in the thesis is the data on total employment from Thomas and Dimsdale (2017) for the 1760-1914 period. The series is extrapolated backwards to 1700 at the growth rate of population. The population data is available from Broadberry *et al*. (2015).

Capital stock

Real capital stock data in 2013 GBP is available from Thomas and Dimsdale (2017) for the period between 1760 and 1914. The data is extrapolated backwards to 1700 using Madsen *et al.*'s (2010) capital stock estimates.

Rental cost of capital

There is no published data on rental cost of capital. Thus, price of capital goods is used to proxy the rental cost of capital. Capital goods price data is available from Feinstein and Pollard (1988) for the period 1851 – 1914. The data is then extrapolated backwards at the growth rate of the cost of capital calculated using $r = P_k r_k$, where P_k is the price of raw materials used to construct/produce capital goods and r_k is nominal return on non-land assets. Following Allen (2009), the prices of iron, bricks and wood are used to compute P_k for the period between 1700 and 1850. Prices of iron, wood and bricks are from Clark (2007). The data on return on non-land assets is available from Clark (1998).

Energy and energy services

Data on energy prices, energy service prices and energy consumption are available from Fouquet (2008; 2011) Energy consumption data is comprising of animal muscle power, wood, coal, crude oil and gas. Aggregate measure of energy price is the average of prices of individual fuels weighted by their share in total energy mix. Energy prices are deflated using Robert Allen's CPI index available at http://www.iisg.nl/hpw/data.php.

Wages

Average real wages are composite series based on English and British wage estimates from various sources. Thomas and Dimsdale (2017) splice the data and construct composite series of average wage for Great Britain.

Factor cost shares

Cost share of labour in Income is taken from Clark (2010) for the 1700 – 1854 period. For the remaining period, the data is obtained from Thomas and Dimsdale (2017) who used Mitchell's (1988) data to construct the shares. The cost share of energy is available from Fouquet (2008) for the entire period and the capital's share is calculated using the data on the shares of labour and energy following Madsen *et al.* (2010).

Appendix B

B.1 Solution to the MLE problem

Differentiating the log-likelihood function 11 with respect to δ yields the following:

$$\frac{\partial l}{\partial \delta} = \sum_{t=1}^{N}(\boldsymbol{y_t} - (\delta - \frac{1}{\phi}\boldsymbol{x_t}))$$

Equating the first order condition to zero and then solving for δ gives the following estimator:

$$\hat{\delta} = \frac{1}{N}\sum_{t=1}^{N}\left(\boldsymbol{y_t} + \frac{1}{\phi}\boldsymbol{x_t}\right) \tag{B.1}$$

in scalar form:

$$\begin{bmatrix}\delta_l\\ \delta_e\end{bmatrix} = \frac{1}{N}\sum_{t=1}^{N}\left(\begin{bmatrix}y_{l,t}\\ y_{e,t}\end{bmatrix} + \frac{1}{\phi}\begin{bmatrix}x_{l,t}\\ x_{e,t}\end{bmatrix}\right)$$

To derive the estimator for the covariance matrix Σ, first, express the log-likelihood function 11 as follows:

$$l(\boldsymbol{y}|\delta,\phi,\Sigma) \propto -\frac{N}{2}\log|\Sigma| - \frac{1}{2}\sum_{t=1}^{N} trace\left[\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right)\Sigma^{-1}\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right)^T\right]$$

And then using the trace rule, $trace[ABC] = trace[BAC]$, yields:

$$= -\frac{N}{2}\log|\Sigma| - \frac{1}{2}trace\left[\Sigma^{-1}\sum_{t=1}^{N}\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right)\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right)^T\right]$$

Using the fact that $-\log|\Sigma| = \log|\Sigma^{-1}|$ and taking the derivative of the function with respect to Σ^{-1}yields:

$$\frac{\partial l}{\partial \Sigma^{-1}} = \frac{N}{2}\Sigma + \frac{1}{2}\sum_{t=1}^{N}\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right)\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)^T\right)$$

Finally, setting the above first order condition to zero gives the maximum likelihood estimator for Σ:

$$\hat{\Sigma} = \frac{1}{N}\sum_{t=1}^{N}\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)\right)\left(\boldsymbol{y_t} - \left(\delta - \frac{1}{\phi}\boldsymbol{x_t}\right)^T\right) \qquad \text{B.2}$$

Now, differentiating (11) with respect to ϕ yields:

$$\frac{\partial l}{\partial \phi} = \sum_{t=1}^{N}(\frac{1}{\phi^2}\boldsymbol{x_t^T})\Sigma^{-1}(\boldsymbol{y_t} - (\delta - \frac{1}{\phi}\boldsymbol{x_t}))$$

Equating the trace of the partial derivative to zero, applying the trace rule and then simplifying yields:

$$trace(\sum_{t=1}^{N}(\frac{1}{\phi^2}\boldsymbol{x}_t^T)\Sigma^{-1}(\boldsymbol{y_t} - (\delta - \frac{1}{\phi}\boldsymbol{x_t}))) = 0$$

$$trace\left(\sum_{t=1}^{N}\begin{bmatrix} y_{l,t} - \left(\delta - \frac{1}{\phi}x_{l,t}\right) \\ y_{e,t} - \left(\delta - \frac{1}{\phi}x_{e,t}\right)\end{bmatrix}\begin{bmatrix}\frac{1}{\phi^2}x_{l,t} & \frac{1}{\phi^2}x_{e,t}\end{bmatrix}\right) = 0$$

$$\sum_{t=1}^{N}(y_{l,t} - (\delta - \frac{1}{\phi}x_{l,t}))\frac{1}{\phi^2}x_{l,t} + \sum_{t=1}^{N}(y_{e,t} - (\delta - \frac{1}{\phi}x_{e,t}))\frac{1}{\phi^2}x_{e,t} = 0$$

Then, solving for ϕ gives the following closed form representation:

$$\hat{\phi} = \frac{\sum_{t=1}^{N}(y_{l,t}^2 + y_{e,t}^2)}{\sum_{t=1}^{N}(\delta_l - y_{l,t})x_{l,t} + \sum_{t=1}^{N}(\delta_e - y_{e,t})x_{e,t}} \qquad \text{B.3}$$

B.2 Bootstrap simulation

The parameters of the log-likelihood function are estimated by the following iterative process:

1. Choose a tolerance limit
2. Make a guess for $\boldsymbol{\delta}$
3. Compute $\hat{\phi}$ using equation B.3
4. Use $\hat{\phi}$ in B.1 to compute a new value for $\boldsymbol{\delta}$
5. Use the updated value of $\boldsymbol{\delta}$ in B.3
6. Repeat steps 3-5 a large number of times until the difference between the subsequent values of $\boldsymbol{\delta}$ equals the tolerance limit set in step 1.
7. Once convergence is achieved, use $\hat{\phi}$ and $\boldsymbol{\delta}$ in B.1 and B.3 to obtain the final solution.